Exponential Nonlinearity and the Method of Symmetrization (Variational Problems and Related Topics)

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数理解析研究所講究録 1307 巻 2003 年 175-188

Exponential Nonlinearity and the Method of Symmetrization

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1 Introduction

The Liouville type equation, that is, a semilinear elliptic equation with an *exponential nonlinearity*

$$-\Delta v = V(x)e^{v}, \tag{1}$$

appears in various areas of mathematical science in two dimensional spaces such as the statistical mechanics of point vortices [2, 3, 9], the prescribed Gaussian curvature problems [7, 4], the Chern-Simons-Higgs gauge theory [19], the stationary problems of chemotaxis [17], and so on.

The purpose of this note is to explain the outline of a new method of *symmetrization* available for the study of the blow-up behaviour in the limit of solution sequences for the Liouville type equation (1) for $V(x) > 0$. See [15] for details.

We developed the method in the study of the Palais-Smale sequence of the functional

$$J_{\lambda}(v) = \frac{1}{2}(\|\nabla v\|_2 + a\|v\|_2) - \lambda \log \fint_{\Omega} K(x)e^{v(x)}dx \quad \text{for} \quad v(x) \in H^1(\Omega),$$

where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary, $\|\cdot\|_p$ is the standard $L^p(\Omega)$ norm, a is a positive constant, $\fint_{\Omega} = \frac{1}{|\Omega|}\int_{\Omega}$, and $K(x)$ is a positive smooth function on $\overline{\Omega}$.

This functional appears in relation to the free energy functional in a model of chemotaxis. The Euler-Lagrange equation of $J_{\lambda}(v)$ is as follows:

$$\begin{cases} -\Delta v + av = \dfrac{\lambda K(x)e^{v}}{\int_{\Omega} K(x)e^{v}dx} & \text{in} \quad \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

This equation was introduced by Childress and Percus [6] as the equilibrium state of the Keller-Segel system [8] of chemotaxis.

Definition 1. *$\{(\lambda_k, v_k)\} \subset \mathbf{R} \times H^1(\Omega)$ is a Palais-Smale sequence of $J_\lambda(\cdot)$ if it satisfies following two properties:*

- *$\lambda_k \geq 0$ and there exists $\lambda_0 \in [0,\infty)$ such that $\lambda_k \longrightarrow \lambda_0$ as $n \longrightarrow \infty$.*
- *$J'_{\lambda_k}(v_k) \longrightarrow 0$ strongly in $H^1(\Omega)'$, where $J'_{\lambda_k}(v_k)$ denotes the Fréchet derivative of $J_{\lambda_k}(\cdot)$ in $H^1(\Omega)$ at v_k.*

The condition $J'_{\lambda_k}(v_k) \longrightarrow 0$ in $H^1(\Omega)'$ is equivalent to the existence of the sequence $\{w_k\} \subset H^1(\Omega)$ such that

$$\begin{cases} -\Delta(v_k - w_k) + a(v_k - w_k) = \dfrac{\lambda_k K(x) e^{v_k}}{\int_\Omega K(x) e^{v_k} dx} & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu}(v_k - w_k) = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

and

$$\|w_k\|_{H^1(\Omega)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{4}$$

Let $u_k = v_k - w_k + a\Delta_D^{-1}(v_k - w_k)$, for example, where $-\Delta_D$ is the Laplace operator in Ω with the Dirichlet boundary condition of $\partial\Omega$. Then easily we are able to see that u_k satisfies

$$-\Delta u_k = V_k e^{u_k} \quad \text{in} \quad \Omega, \tag{5}$$

where

$$V_k(x) = \frac{\lambda_k K(x) e^{w_k - a\Delta_D^{-1}(v_k - w_k)}}{\int_\Omega K(x) e^{v_k} dx}.$$

Accordingly, we are able to reduce the study of the Palais-Smale sequences of the functional $J_\lambda(\cdot)$ to the study of the solution sequences of families of the Liouville type equations and we are able to use various methods developed for (5).

The difficulty in the analysis of the Palais-Smale sequence of $J_\lambda(\cdot)$ stems from the fact that $J_\lambda(\cdot)$ is *not* coercive on $H^1(\Omega)$ in general, that is, we are not always able to control the behaviour of $H^1(\Omega)$ norm for the Palais-Smale sequence.

We note that, when $\lambda < 4\pi$, we are able to see that $J_\lambda(\cdot)$ is coercive from the Chang-Yang inequality [5], which is $H^1(\Omega)$ counterpart of the Trudinger-Moser inequality in $H_0^1(\Omega)$ [12]. Indeed, we have

$$\frac{1}{2}\|\nabla w\|_2^2 + 4\pi \fint_\Omega w - 4\pi \log \fint_\Omega e^w \geq -C, \tag{6}$$

where $C > 0$ is a constant determined by Ω. It follows that $J_\lambda(\cdot)$ is coercive when $\lambda < 4\pi$.

Except for the case $\lambda < 4\pi$, we must study the behaviour of Palais-Smale sequence from the only a priori bound on the right-hand side of (3) (or (5))

$$\int_\Omega \frac{\lambda_k K(x)e^{v_k}}{\int_\Omega K(x)e^{v_k}dx}dx \left(= \int_\Omega V_k e^{u_k}dx\right) = \lambda_k = O(1)$$

and the special characteristics of the *exponential nonlinearity.*

Using the result of Brezis and Merle [1] for (1) carefully, we are able to obtain the following *rough* estimate:

Theorem 1 (Rough estimate). *Let $\{(\lambda_k, v_k)\}$ be a Palais-Smale sequence of $J_\lambda(\cdot)$ and put*

$$\mu_k(dx) = \frac{\lambda_k K(x)e^{v_k}}{\int_\Omega K(x)e^{v_k}dx}dx.$$

Taking a subsequence if necessary, we may assume that

$$\mu_k(dx) \rightharpoonup \mu(dx) \quad *\text{-weakly in} \quad M(\overline{\Omega}),$$

where $M(\overline{\Omega}) = C(\overline{\Omega})'$ denotes the space of signed measures on the compact set $\overline{\Omega}$. Then the following alternative holds:

> *(i)(compactness) there exists $v \in H^1(\Omega)$ and a further subsequence of $\{v_k\}$ such that $v_k \longrightarrow v$ strongly in $H^1(\Omega)$ and*
>
> $$\mu(dx) = \frac{\lambda_0 K(x)e^{v}}{\int_\Omega K(x)e^{v}dx}dx,$$

or

> *(ii)(concentration) there exists a non-empty set $\mathcal{S} \subset \overline{\Omega}$ and the positive number $m(x_0)$ for each $x_0 \in \mathcal{S}$ such that*
>
> $$m(x_0) \geq \begin{cases} 4\pi & \text{for } x_0 \in \mathcal{S} \cap \Omega, \\ 2\pi & \text{for } x_0 \in \mathcal{S} \cap \partial\Omega, \end{cases} \tag{7}$$
>
> *and $\mu(dx) = \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx)$. We note that $\lambda_0 \geq 2\pi$ and $\#\mathcal{S} < \infty$ follow in this case.*

Using *the method of symmetrization* in this note, we are able to refine the above theorem as follows:

Theorem 2 (Fine estimate). *Let $\{(\lambda_k, v_k)\}$ be a Palais-Smale sequence of $J_\lambda(\cdot)$ and let $\{w_k\} \subset H^1(\Omega)$ be a sequence of functions satisfying (3) and (4). Moreover, suppose $\{w_k\} \subset W^{1,\infty}(\Omega)$ and*

$$\|w_k\|_{W^{1,\infty}(\Omega)} \longrightarrow 0 \quad as \quad n \longrightarrow \infty. \tag{8}$$

Then $m(x_0)$ in the conclusion (ii) of Theorem 1 is improved as

$$m(x_0) = \begin{cases} 8\pi & \text{for } x_0 \in \mathcal{S} \cap \Omega, \\ 4\pi & \text{for } x_0 \in \mathcal{S} \cap \partial\Omega. \end{cases} \tag{9}$$

Especially, we have $\lambda_0 \in 4\pi\mathbf{N}$ and $2\#(\mathcal{S} \cap \Omega) + \#(\mathcal{S} \cap \partial\Omega) = \lambda_0/(4\pi)$. Furthermore, we have

$$\nabla_x \left(m(x_0)H(x, x_0) + \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m(x_0')G(x, x_0') + \log K(x) \right)\Bigg|_{x=x_0} = 0 \tag{10}$$

for each $x_0 \in \mathcal{S}$, where $G(x, y)$ is the Green function of $-\Delta + a$ with the Neumann boundary condition and

$$H(x, y) = \begin{cases} G(x, y) - \frac{1}{2\pi} \log|x - y|^{-1} & \text{for } y \in \Omega, \\ G(x, y) - \frac{1}{\pi} \log|x - y|^{-1} & \text{for } y \in \partial\Omega. \end{cases}$$

In (10), ∇_x takes only tangential derivative in the case of $x_0 \in \partial\Omega$.

We know similar results to Theorem 2 on the *quantization* phenomenon for the blow-up sequences of solutions for (5) [10, 11]. We note that the assumption (4) is also too naive to apply them to the study of the Palais-Smale sequences of $J_\lambda(\cdot)$.

2 The method of symmetrization

We see the idea of the method of symmetrization in the sketch of the proof of Theorem 2. For simplicity, we assume that

$$K(x) \equiv 1.$$

2.1 Unfolding the exponential nonlinearity

Let

$$u_k = \frac{\lambda_k e^{v_k}}{\int_\Omega e^{v_k} dx}. \tag{11}$$

Then the equation (3) is reduced to

$$\begin{cases} -\Delta(v_k - w_k) + a(v_k - w_k) = u_k \quad \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu}(v_k - w_k) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{12}$$

Using (6) for $v_k \in H^1(\Omega)$, we obtain that

$$\nabla u_k \left(= \frac{\lambda_k e^{v_k}}{\int_\Omega e^{v_k} dx} \nabla v_k \right) = u_k \nabla v_k \in L^q(\Omega) \quad \text{for } 1 \leq q < 2, \tag{13}$$

especially, $u_k \in W^{1,q}(\Omega)$ for $1 \leq q < 2$.

Applying the divergence operator $\nabla \cdot$ to both sides of (13), we obtain that

$$0 = \nabla \cdot (\nabla u_k - u_k \nabla v_k) \quad \text{in} \quad \mathcal{D}', \tag{14}$$

Coupling (12) and (14), we obtain the system of equations for u_k and v_k instead of the scalar equation (3).

We compare this system of equations with the following stationary Keller-Segel model of chemotaxis:

$$\begin{cases} 0 = \nabla \cdot (\nabla u - u \nabla v) \quad \text{in} \quad \Omega, \\ -\Delta v + av = u \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{15}$$

For this system of equation, the case $u \geq 0$ is important because u mean the density of cellular slime molds (and v means the concentration of chemical substances secreted by themselves).

It should be remarked that any non-negative solution u for (15), which is positive in $\overline{\Omega}$ from the maximum principle, satisfies

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx}, \tag{16}$$

where λ is a positive constant and v is a solution of

$$\begin{cases} -\Delta v + av = \dfrac{\lambda e^v}{\int_\Omega e^v dx} \quad \text{in} \quad \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{17}$$

which is (2) for our case $K(x) \equiv 1$. We are able to see this fact as follows.

Writing the first equation of (15) as

$$0 = \nabla \cdot u\nabla(\log u - v),$$

we obtain

$$\int_\Omega u|\nabla(\log u - v)|^2 dx = 0,$$

that is,

$$\log u - v = C$$

for some constant C. In terms of $\lambda = \|u\|_1$, this relation is transformed as (16).

From these argument, we may say that the first equation of (15) is *folded* in the exponential relation (16). Thus we may say that we *unfolded* the exponential nonlinearity of (3) in this subsection.

2.2 Reduction of the system to the scalar equation

Let $(-\Delta + a)_N^{-1}$ be the inverse operator of $-\Delta + a$ under the Neumann boundary condition. Then we are able to write the equation (12) as

$$v_k = (-\Delta + a)_N^{-1} u_k + w_k.$$

Thus, from (14), we have the equation of u_k

$$0 = \nabla \cdot (\nabla u_k - u_k \nabla\{(-\Delta + a)_N^{-1} u_k + w_k\}), \tag{18}$$

that is, for every test function $\psi \in C^2(\overline{\Omega})$ satisfying $\partial\psi/\partial\nu = 0$ on $\partial\Omega$,

$$\begin{aligned} -\int_\Omega u_k \nabla w_k \cdot \nabla\psi dx = & \int_\Omega u_k \Delta\psi dx \\ & + \int_\Omega \int_\Omega \nabla_x G(x,y) \cdot \nabla\psi(x) u_k(x) u_k(y) dx dy, \end{aligned} \tag{19}$$

where $G(x,y)$ is the Green function for $(-\Delta + a)_N^{-1}$.

We obtain Theorem 2 from the study of the limit of (19) as $k \longrightarrow \infty$ for an appropriate test function ψ.

It is well known that

$$G(x,y) = \frac{1}{2\pi} \log|x-y|^{-1} + H(x,y), \tag{20}$$

where $H(x,y) \in C^{1,\theta}(\Omega \times \Omega)$ for every $0 < \theta < 1$. We note that $\nabla_x G(x,y)$ is singular at the diagonal set $\{(x,y); x = y\}$ of $\Omega \times \Omega$.

From this singularity, we are not able to know the limit of the term

$$\int_\Omega \int_\Omega \nabla_x G(x,y) \cdot \nabla\psi(x) u_k(x) u_k(y) dx dy \tag{21}$$

in this really style, since $u_k(x)u_k(y)dxdy$ concentrates on the diagonal set of $\Omega \times \Omega$.

The method of *symmetrization* is a technique that we use to avoid this difficulties.

2.3 Symmetrization of the equation

We note that the Green function $G(x,y)$ for the operator $(-\Delta + a)_N^{-1}$ is symmetric, that is,

$$G(x,y) = G(y,x).$$

Thus, we have

$$\begin{aligned} (21) &= \int_\Omega \int_\Omega \nabla_y G(x,y) \cdot \nabla\psi(y) u_k(x) u_k(y) dx dy \\ &= \frac{1}{2} \int_\Omega \int_\Omega \rho_\psi(x,y) u_k(x) u_k(y) dx dy, \end{aligned} \tag{22}$$

where

$$\rho_\psi(x,y) = \nabla_x G(x,y) \cdot \nabla\psi(x) + \nabla_y G(x,y) \cdot \nabla\psi(y).$$

This is the idea of *symmetrization.* Using this expression, we see the sketch of the proof of Theorem 2 for the case $\mathcal{S} = \{0\} \subset \Omega$ in the next subsection.

It should be remarked that Senba and Suzuki [18] used this method of symmetrization in the study of the weak solution of the Nagai model [13]

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) \quad \text{in} \quad \Omega \times (0,T), \\ -\Delta v + av = u \quad \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \end{cases} \tag{23}$$

which is a time-dependent version of (15) and a simplified version of the Keller-Segel model of chemotaxis.

Senba and Suzuki [18] proved that, in spite of the singularity of the Green function $G(x,y)$, the symmetrized kernel $\rho_\psi(x,y) \in L^\infty(\Omega \times \Omega)$ for every $\psi \in C^2(\overline{\Omega})$ satisfying $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$. We note that this fact is not enough for our analysis because u_k converges to a sum of Dirac measures in $M(\overline{\Omega})(= C(\overline{\Omega})')$ from Theorem 1. Thus we must choose a more special test function ψ.

2.4 The good test function and the limit

Here we see the sketch of the proof of Theorem 2 for the case $\mathcal{S} = \{x_0\} \subset \Omega$, that is,

$$\mu_k(dx) = u_k dx = \frac{\lambda_k e^{v_k}}{\int_\Omega e^{v_k} dx} dx \longrightarrow m(x_0)\delta_{x_0}(dx), \in M(\overline{\Omega})$$

where $m(x_0)$ is a constant satisfying $m(x_0) \geq 4\pi$. Then what we have to prove is that

$$m(x_0) = 8\pi \quad \text{and} \quad \nabla_x H(x, x_0)\big|_{x=x_0} = 0. \tag{24}$$

We discuss general cases in the next section.

Here we note that, in the course of the proof of Theorem 1, we have

$$u_k \longrightarrow 0 \quad \text{in} \quad L^p(\Omega \backslash B_\varepsilon(x_0))$$

for every $1 \leq p < \infty$ and $0 < \varepsilon \ll 1$, which we use in the rigorous treatment of the following sketch of the proof of Theorem 2.

We divide $\rho_\psi(x, y)$ into two part:

$$\rho_\psi(x, y) = (\mathrm{I}) + (\mathrm{II}),$$

where

$$\begin{aligned}
(\mathrm{I}) &= \nabla_x\left(\frac{1}{2\pi}\log|x-y|^{-1}\right)\cdot\nabla\psi(x) + \nabla_y\left(\frac{1}{2\pi}\log|x-y|^{-1}\right)\cdot\nabla\psi(y), \\
&= -\frac{1}{2\pi}\cdot\frac{(\nabla\psi(x)-\nabla\psi(y))\cdot(x-y)}{|x-y|^2}, \\
(\mathrm{II}) &= \nabla_x H(x,y)\cdot\nabla\psi(x) + \nabla_y H(x,y)\cdot\nabla\psi(y).
\end{aligned}$$

Now we take

$$\psi(x) = |x - \mathbf{a}|^2 \varphi(x)$$

for $\mathbf{a} \in \mathbf{R}^2$ and $\varphi(x) \in C_0^2(\Omega)$ satisfying $\varphi(x) \equiv 1$ near $x_0 \in \Omega$. We note that this ψ satisfies the boundary condition $\frac{\partial\psi}{\partial\nu} = 0$.

For this ψ, we have

$$\nabla\psi = 2(x - \mathbf{a}) \quad \text{and} \quad \Delta\psi = 4 \quad \text{near} \quad x_0 \in \Omega. \tag{25}$$

Moreover

$$\begin{aligned}
(\mathrm{I}) &= -\frac{1}{2\pi}\cdot\frac{\{2(x-\mathbf{a}) - 2(y-\mathbf{a})\}\cdot(x-y)}{|x-y|^2} \\
&= -\frac{1}{\pi}\cdot\frac{(x-y)\cdot(x-y)}{|x-y|^2} \\
&\equiv -\frac{1}{\pi} \quad \text{near} \quad x_0 \in \Omega.
\end{aligned}$$

Thus we have

$$\int_\Omega \int_\Omega \text{(I)}\ u_k(x)u_k(y)dxdy \longrightarrow -\frac{m(x_0)^2}{\pi} \quad \text{as} \quad k \longrightarrow \infty.$$

On the other hand, since (II) is continuous near (x_0, x_0), we have

$$\begin{aligned}&\int_\Omega \int_\Omega \text{(II)}\ u_k(x)u_k(y)dxdy \\ &\qquad \longrightarrow m(x_0)^2 \{\nabla_x H(x_0,x_0) \cdot 2(x_0 - \mathbf{a}) + \nabla_y H(x_0,x_0) \cdot 2(x_0 - \mathbf{a})\} \\ &\qquad = 4m(x_0)^2 (x_0 - \mathbf{a}) \cdot \nabla_x H(x, x_0)|_{x=x_0} \quad \text{as} \quad k \longrightarrow \infty.\end{aligned}$$

From these calculation, we are able to see the limit of the term (21). We see the limits of the other terms in the equation (19).
Using (25), we have

$$-\int_\Omega u_k \Delta\psi \longrightarrow -m(x_0)\Delta\psi(0) = -4m(x_0).$$

On the other hand, as we assumed (8), we have

$$\int_\Omega u_k \nabla w_k \cdot \nabla\psi \longrightarrow 0$$

since $\|u_k\|_1 = \lambda_k = O(1)$.

Combining these calculations, we have the following equation as a limit of the equation (19):

$$-4m(x_0) = \frac{m(x_0)^2}{2\pi} + 2m(x_0)^2(x_0 - \mathbf{a}) \cdot \nabla_x H(x, x_0)|_{x=x_0} \tag{26}$$

for every $\mathbf{a} \in \mathbf{R}^2$.

Letting $\mathbf{a} = x_0$ in (26), we have

$$-4m(x_0) = -\frac{m(x_0)^2}{2\pi}, \quad \text{that is,} \quad m(x_0) = 8\pi$$

since $m(x_0) \geq 4\pi > 0$ from Theorem 1. Then, taking $x_0 - \mathbf{a} = \nabla_x H(x, x_0)|_{x=x_0}$ in (26), we have

$$\nabla_x H(x, x_0)|_{x=x_0} = 0.$$

Thus we obtain (24).

3 Various remarks

3.1 When $K(x) \not\equiv 1$

Also in this case, we set

$$u_k = \frac{\lambda_k K(x) e^{v_k}}{\int_\Omega K(x) e^{v_k} dx}$$

as we did in (11).

We are able to write

$$u_k = \frac{\lambda_k e^{v_k + \log K(x)}}{\int_\Omega K(x) e^{v_k} dx},$$

since we assumed $K(x) > 0$ in $\overline{\Omega}$. Accordingly, we have

$$\nabla u_k = u_k \nabla \{v_k + \log K(x)\}$$

instead of (13) and

$$0 = \nabla \cdot [\nabla u_k - u_k \nabla \{v_k + \log K(x)\}]$$

instead of (14).

Thus we need to add

$$\int_\Omega u_k \nabla \log K(x) \cdot \nabla \psi dx$$

to the right-hand side of (19) and

$$2m(x_0)(x_0 - \mathbf{a}) \cdot \nabla \log K(x)|_{x=x_0}$$

to the limit equation (26).

Consequently, we have the term $\log K(x)$ in (10).

3.2 When we have many points in $\mathcal{S}$

We fix $x_0 \in \mathcal{S}$ and assume $\mathcal{S} \backslash \{x_0\} \neq \emptyset$. Moreover we assume $x_0 \in \Omega$ for simplicity.

Then we are able to choose the test function ψ in section 2.4 satisfying that $\operatorname{supp} \psi \cap \mathcal{S} = \{x_0\}$. Moreover, let $\xi \in C_0^2(\Omega)$ be a cut-off function around x_0 satisfying

$$0 \leq \xi(x) \leq 1 \quad \text{in} \quad \Omega \quad \text{and} \quad \xi(x) \equiv 1 \quad \text{in} \quad \operatorname{supp} \psi.$$

We note that

$$\psi = \xi\psi \quad \text{and} \quad \nabla\psi = \xi\nabla\psi.$$

We also note that

$$u_k^0 dx := \xi(x)u_k(x)dx \longrightarrow m(x_0)\delta_{x_0}(dx) \quad * \text{ weakly in } M(\overline{\Omega}),$$
$$(1-\xi(x))u_k(x)dx \longrightarrow \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m(x_0')\delta_{x_0'}(dx) \quad * \text{ weakly in } M(\overline{\Omega}).$$

We have

$$\begin{aligned}
&\int_\Omega \int_\Omega \nabla_x G(x,y) \cdot \nabla\psi(x) u_k(x) u_k(y) dx dy \\
&\quad = \int_\Omega \int_\Omega \nabla_x G(x,y) \cdot \nabla\psi(x) u_k^0(x) u_k(y) dx dy \\
&\quad = \int_\Omega \int_\Omega \nabla_x G(x,y) \cdot \nabla\psi(x) u_k^0(x) u_k^0(y) dx dy, \\
&\qquad + \int_\Omega \int_\Omega \nabla_x G(x,y) \cdot \nabla\psi(x) u_k^0(x)(1-\xi(y)) u_k(y) dx dy \\
&\quad =: (\mathrm{III}) + (\mathrm{IV}).
\end{aligned} \tag{27}$$

For (III), we have the same limit as (21). On the other hand, since $G(x,y)$ is smooth on $\operatorname{supp}\psi \times \operatorname{supp}(1-\xi)$, we have

$$\begin{aligned}
(\mathrm{IV}) &\longrightarrow m(x_0) \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m(x_0') \nabla\psi(x_0) \cdot \nabla_x G(x_0, x_0') \\
&= 2m(x_0)(x_0 - \mathbf{a}) \cdot \nabla_x \left(\sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m(x_0') G(x, x_0') \right) \Bigg|_{x=x_0} \qquad \text{as} \quad n \longrightarrow \infty.
\end{aligned} \tag{28}$$

Thus we add this limit to the right-hand side of the limit equation (26).

Consequently, we have the term $\sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m(x_0')G(x,x_0')\Big|_{x=x_0}$ in (10).

We end this subsection with some remarks on the case $x_0 \in \partial\Omega$ briefly. In this case, we may flatten the boundary near x_0 conformally and extend the function $v_k - w_k$ as an even function using the Neumann boundary condition in (3). Then we are able to consider the concentration point x_0 as an interior point of an extended domain and the similar argument to this subsection is applicable. We note that the mass 4π of concentration at boundary is a half of 8π. This comes from the above even extension of $v_k - w_k$.

3.3 The symmetrization in other equations

It should be remarked that we know similar method of symmetrization in the two dimensional Euler equation for incompressible ideal fluid.

Let Ω be a simply connected bounded domain with a smooth boundary. Then, in terms of the scalar vorticity field $\omega(x,t)$ and the stream function $\psi(x,t)$, the Euler equations for the incompressible homogeneous ideal fluid with unit density are written as follows:

$$\begin{cases} \omega_t = \nabla \cdot (-\omega \nabla^{\perp} \psi) & \text{in} \quad \Omega \times (0,T), \\ -\Delta \psi = \omega & \text{in} \quad \Omega \times (0,T), \\ \psi = 0 & \text{on} \quad \partial\Omega \times (0,T). \end{cases} \tag{29}$$

We note that the velocity field u is determined by the stream function as

$$u = \nabla^{\perp} \psi = \left(\frac{\partial}{\partial x^2} \psi, -\frac{\partial}{\partial x^1} \psi \right).$$

We also note that

$$\omega = \operatorname{curl} u(x,t) \left(= \frac{\partial}{\partial x^1} u^2(x,t) - \frac{\partial}{\partial x^2} u^1(x,t) \right) = -\Delta \psi.$$

It seems interesting that the equation (29) resembles the Nagai model (23). Moreover, we know the similar method of symmetrization for the weak formulation of the term

$$\nabla \cdot (-\omega \nabla^{\perp} \psi) = \nabla \cdot (-\omega \nabla^{\perp} (-\Delta)_D^{-1} \omega),$$

where $(-\Delta)_D^{-1}$ is the inverse operator of $-\Delta$ with the Dirichlet boundary condition. See [16]. See also [20, 14].

4 Concluding remark

In this note, we see the method of symmetrization through the study of the behaviour of the Palais-Smale sequence for $J_\lambda(\cdot)$. The key of the idea is to unfold the exponential nonlinearity, which is also applicable for general Liouville type equations (1) for $V(x) > 0$. We are now in preparation for such generalization and further application of this method.

References

[1] H. BREZIS AND F. MERLE, *Uniform estimates and blow-up behavior for solutions of* $-\Delta = V(x)e^u$ *in two dimensions*, Comm. in Partial Differential Equations, 16 (1991), pp. 1223–1253.

[2] E. CAGLIOTI, P. L. LIONS, C. MARCHIORO, AND M. PULVIRENTI, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description*, Comm. Math. Phys., 143 (1992), pp. 501–525.

[3] ——, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description. part II*, Comm. Math. Phys., 174 (1995), pp. 229–260.

[4] S. Y. A. CHANG AND P. C. YANG, *Prescribing Gaussian curvature on* $\mathbf{S}^2$, Acta Math., 159 (1987), pp. 215–259.

[5] ——, *Conformal deformation of metric on* $\mathbf{S}^2$, J. Differential Geom., 27 (1988), pp. 259–296.

[6] S. CHILDRESS AND J. K. PERCUS, *Nonlinear aspects of chemotaxis*, Math. Biosci., 56 (1981), pp. 217–237.

[7] J. KAZDAN AND F. WARNER, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2), 99 (1974), pp. 14–47.

[8] E. F. KELLER AND L. A. SEGEL, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., 26 (1970), pp. 399–415.

[9] M. K. H. KIESSLING, *Statistical mechanics of classical particles with logarithmic interactions*, Comm. Pure Appl. Math., 46 (1993), pp. 27–56.

[10] Y. Y. LI, *Harnack type inequality: the method of moving planes*, Comm. Math. Phys., 200 (1999), pp. 421–444.

[11] Y. Y. LI AND I. SHAFRIR, *Blow-up analysis for solutions of* $-\Delta u = Ve^u$ *in dimension two*, Comm. Math. Phys., 43 (1994), pp. 1255–1270.

[12] J. MOSER, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J., 20 (1971), pp. 1077–1092.

[13] T. NAGAI, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl., 5 (1995), pp. 581–601.

[14] H. Ohtsuka, *On the evolution of a high-energy vorticity in an ideal fluid*, Kyushu J. Math., 53 (1999), pp. 37–58.

[15] H. Ohtsuka and T. Suzuki, *Palais-Smale sequence relative to the Trudinger-Moser inequality.* to appear in Calc. Var. Partial Differential Equations.

[16] S. Schochet, *The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation*, Comm. Partial Differential Equations, 20 (1995), pp. 1077–1104.

[17] T. Senba and T. Suzuki, *Some structures of the solution set for a stationary system of chemotaxis*, Adv. Math. Sci. Appl., 10 (2000), pp. 191–224.

[18] ——, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology*, Adv. Differential Equations, 6 (2001).

[19] G. Tarantello, *Multiple condensate solutions for the Chern-Simons-Higgs theory*, J. Math. Phys., 37 (1996), pp. 3769–3796.

[20] B. Turkington, *On the evolution of a concentrated vortex in an ideal fluid*, Arch. Rational Mech. Anal., 97 (1987), pp. 75–87.